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High dimensional behavior of the Kardar-Parisi-Zhang growth dynamics

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We investigate analytically the large dimensional behavior of the Kardar-Parisi-Zhang (KPZ) dynamics of surface growth using a recently proposed nonperturbative renormalization for self-affine surface dynamics. Within this framework, we show that the roughness exponent α decays not faster than $\alpha \sim 1/d$ for large d. This implies the absence of a finite upper critical dimension. [S1063-651X(98)50611-3]

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The study of the nonequilibrium dynamics of rough surfaces and interfaces has received a great deal of attention in the last years [1,2]. Both theoretically and experimentally many efforts have been devoted to single out the traits and features shared by apparently different phenomena. In this context, by analogy with equilibrium statistical mechanics, the search for universality classes is a central task. The Kardar-Parisi-Zhang equation (KPZ) [3] is, for surface growth, the main contribution in this direction. It is the minimal Langevin equation capturing the essence of many different growth models beyond the Gaussian linear theory [1,2,4]. It reads

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t), \tag{1}$$

where $h(x,t)$ is the surface profile, x is the position in a d-dimensional substrate, η is a Gaussian white noise, ν and λ are constants. The KPZ equation also describes the behavior of directed polymers in random media [1], systems with multiplicative noise [5], and it is related to the Burgers equation [6].

A central quantity of interest is the roughness $W(L)$ of a system of linear size L, defined as

$$W^2(L) = \frac{1}{L^d} \sum_x [h(x,t) - \overline{h}]^2, \tag{2}$$

where $\overline{h} = (1/L^d)\Sigma_x h(x,t)$. In many seemingly unrelated growth processes the large scale properties of the roughness are observed to be scale invariant and universal; for example, in the stationary state $W(L) \sim L^{\alpha}$ and correlations decay on a typical time $t_s \sim L^z$, with universal exponents α and z. These critical exponents are not independent, as a consequence of the Galilean invariance of the related Burgers equation [1,6] $\alpha + z = 2$. It is thus sufficient to focus the attention on one exponent, say α.

The theoretical analysis of the KPZ is extremely difficult. Apart from the $d=1$ case, where a special symmetry makes an exact solution possible with $\alpha = 1/2$, the situation is still quite controversial despite the large effort devoted to the problem. In particular, the fundamental issue of the existence of an upper critical dimension d_c, above which the exponents recover their mean-field (or infinite dimensional) values ($\alpha = 0$) [7], is highly debated [8]. The application of field theoretical tools presents an inherent problem: one indeed finds that the fixed point controlling the rough phase of the KPZ is not accessible to perturbation expansion in λ; this fact renders standard field theoretical tools inadequate for this problem. Early applications of nonperturbative methods such as functional renormalization group [9] and Flory-type arguments [10] suggested that $d_c = 4$, in agreement with a $1/d$ expansion [11] around the $d = \infty$ limit. Later the mode-coupling approximation led to contradictory results suggesting the existence of a finite d_c [12] or $d_c = \infty$ [13]. Arguments for a finite d_c based on directed [14] or invasion [15] percolation have also been proposed. More recently a detailed analysis of a $d = 2 + \epsilon$ perturbative expansion revealed a singularity at $d = 4$ [16], leading Lässig to the conclusion that $d_c = 4$ is the upper critical dimension of the KPZ dynamics [17].

Numerical simulations of models in the KPZ universality class markedly disagree with this last conclusion [8], showing that $\alpha > 0$ at least up to $d = 7$ [18]. In particular, numerical results suggest a large-d behavior $\alpha \sim 1/d$ in agreement with early conjectures (See Ref. [2], p. 75). Both of these conclusions were confirmed by a recently proposed renormalization group (RG) approach [19]. The key idea of this approach is that the geometric scaling of the growing surface can be ascribed to a scale invariant dynamic process, which builds the same correlations at all length scales. This *scale invariant dynamics* is the fixed point of the RG transformation, which is derived by consistency requirements of the description of the same system at two different scales. Analogous ideas, implemented *via* a real space RG, have proved to be quite powerful to investigate the critical properties of nonequilibrium, strong coupling problems [20]. The implicit nature of the RG transformation, which is similar in spirit to the idea of phenomenological RG [21], allows us to avoid the use of hierarchical lattices, a source of incontrollable approximations, specially in high dimensions. Remarkably, the exponents predicted by the RG are in excellent agreement with numerical simulations up to $d = 7$.

In this Rapid Communication we analyze the large-d behavior of this RG approach and show that it predicts that the

roughness exponent α vanishes not faster than $1/d$ for $d \gg 1$. This rules out the existence of a finite upper critical dimension. In what follows we expose the essential concepts of this method and apply it to the analytical study of the KPZ dynamics in the large-d limit.

Consider a growing surface, whose dynamics, at the microscopic scale, is defined in terms of a stochastic equation, such as Eq. (1), or by a discrete model. If we partition the $(d+1)$-dimensional space in cells of lateral size $L_k = 2^k L_0$ and vertical size h_k, we obtain a *static* description of the surface at the coarse-grained scale L_k: With some majority rule each block is declared to be empty or filled. For each substrate cell i the number $h(i)$ of filled blocks on top of it identifies the interface configuration, in units h_k, at scale L_k. Note that h_k is an independent parameter of the static description. Scale invariance implies that if h_k is properly chosen, the coarse-grained system looks similar at all (large enough) length scales L_k. The optimal geometric description, which best exhibits scale invariance, in our case, is that with $h_k \propto W(L_k) \sim L_k^{\alpha}$ of the same order of typical height fluctuations over a distance L_k. In the RG procedure, we shall fix $h_k = 2W(L_k)$ in order to have a scale invariant description of the surface (see Ref. [19] for details). The coarse-graining procedure, which defines the static description in terms of blocks of size L_k, also induces a flow of the microscopic dynamics towards an *effective dynamics* at the same scale L_k; this is defined in terms of the transition rates for the addition of an occupied block. The main feature of KPZ dynamics is lateral growth [3,4], and this suggests the following minimal parametrization of the growth rates at the generic scale L_k is

$$r[h(i) \to h(i)+1] \equiv 1 + x_k \sum_{j \text{nn} i} \max[0, h(j) - h(i)]. \quad (3)$$

The first term is the contribution of the vertical growth (i.e., random deposition) and the second term is the contribution of lateral growth. Indeed the sum over neighbor block sites j counts the area of the vertical surface exposed towards site i. x_k is then the ratio of lateral to vertical growth rates. We shall come back later, in the conclusions, to the approximations implied by Eq. (3).

In order to derive the RG transformation, let us partition a system of size L into ℓ^d cells of size L/ℓ. We observe that the roughness $W^2(L)$ can be written as the average roughness $W^2(L/\ell)$ *inside* single cells plus the fluctuations of the average height *among* different cells. The second contribution, within the description at scale L/ℓ, is simply given by the roughness $\omega^2(\ell, x)$ of a system of ℓ cells, with x being the dynamic parameter at scale L/ℓ, times the square height of the cells $[2W(L/\ell)]^2$. Hence we find

$$W^2(L) = W^2(L/\ell)[1 + 4\omega^2(\ell, x)], \quad (4)$$

which is the basis of the RG approach. With $\ell = 4$, $L = L_{k+2}$ and $x = x_k$, it gives $W^2(L_{k+2})/W^2(L_k)$. The same quantity can be alternatively computed using Eq. (4) with $\ell = 2$, once with $L = L_{k+2}$ and $x = x_{k+1}$, and a second time with $L = L_{k+1}$ and $x = x_k$. The consistency of the two calculations yields an implicit RG transformation

$$1 + 4\omega^2(2, x_{k+1}) = \frac{1 + 4\omega^2(4, x_k)}{1 + 4\omega^2(2, x_k)} \quad (5)$$

for the dynamic parameter x_k. The *attractive* fixed point $x^* = \lim_{k\to\infty} x_k$ (if it exists) identifies the *scale invariant dynamics* and Eq. (4) with $x_k = x^*$ finally yields the roughness exponent

$$\alpha = \lim_{k\to\infty} \log_2 \sqrt{\frac{W^2(L_{k+1})}{W^2(L_k)}} = \frac{\log[1 + 4\omega^2(2, x^*)]}{2 \log 2}. \quad (6)$$

Equations (5) and (6) are the starting point of our analysis. A more detailed discussion of their derivation can be found in Ref. [19]. We note here that the existence of an attractive fixed point x^* implies that the process is ''self-organized'': No fine tuning is necessary in order to observe the critical behavior.

A key observation is that, since $\omega^2(\ell, x) \to 0$ for $x \to \infty$, $x^* = \infty$ is a fixed point of Eq. (5) corresponding to $\alpha = 0$. Therefore the RG scheme allows, in principle, for the occurrence of a finite upper critical dimension d_c ($\alpha = 0$ for $d \geqslant d_c$) and the existence of a finite attractive fixed point for all d is a nontrivial prediction. A finite stable fixed point was found in Ref. [19] for $d = 1, \ldots, 8$ using Monte Carlo methods to compute $\omega^2(\ell, x)$. The same method was also applied to the Gaussian theory [$\lambda = 0$ in Eq. (1)], recovering the result $d_c = 2$, i.e., $\alpha = 0$ for $d \geqslant 2$ [22]. Though very powerful, the Monte Carlo method cannot be pushed to very high dimensions, nor does it provide an explicit analytic behavior of α as a function of d.

In the following we study analytically the large-d limit of the RG in order to extract its predictions on the existence of a finite upper critical dimension and on the large-d behavior of the roughness exponent.

The technical difficulty lies in the explicit calculation of the functions $\omega^2(\ell, x)$ for $\ell = 2$, 4. For $d \gg 1$ we expect $\alpha \ll 1$, which means that surface fluctuations $\omega(\ell, x) \sim \ell^{\alpha} \simeq 1 + \alpha \ln \ell + \ldots$ are of order 1. This suggests that for a system of small size ℓ we can reasonably account for the fluctuations of the interface if we allow $h(i)$ to take only two values: $h(i) = h_0$ or $h(i) = h_0 + 1$. This drastic approximation has the advantages of making the explicit computation feasible on the one hand, and of providing a lower bound for the exponent α on the other. We shall come back later to this important issue. Let us only stress, for the time being, that a lower bound on α is sufficient to exclude the existence of a finite d_c.

In the above approximation, growth can only occur on ''low'' sites [$h(i) = h_0$]. This means that Eq. (3) is only valid if $h(i) = h_0$ and the rates vanish on ''high'' sites [$h(i) = h_0 + 1$]. It is convenient to classify the possible configurations $\{h(i)\}$ by the number n of ''high'' sites. The roughness Eq. (2) of each configuration of n ''high'' sites is the same and is equal to $(1 - n/\ell^d) n/\ell^d$ and the dynamics involves only transitions from configurations with n to configurations with $n+1$ ''high'' sites. We can then group all configurations $\{h(i)\}$ with n ''high'' sites in the same effective state with a great simplification of the structure of the master equation (the state with $n = \ell^d$ is equivalent to the flat surface $n = 0$). The only non-vanishing transition rates $r(n$

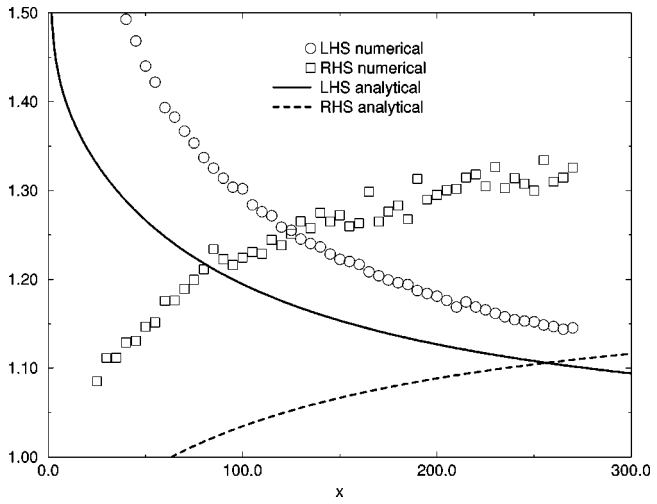

FIG. 1. Graphic analysis of Eq. (5) from the present approximation and from Monte Carlo evaluation for $d=7$.

$\rightarrow n+1$) are obtained from Eq. (3) summing on all possible final configurations and taking the average on the initial configurations, which leads to

$$r(n\rightarrow n+1)=\ell^d-n+x\,\Omega_n\,. \tag{7}$$

The first term here accounts for vertical growth, which can occur only on the ℓ^d-n ''low'' sites. The second term is the contribution of lateral growth and Ω_n is the average number of lateral walls (i.e., the surface between ''low'' and ''high'' sites) in configurations with n ''high'' sites. Assuming that ''low'' and ''high'' sites are randomly distributed, each ''low'' site has on average $2dn/\ell^d$ ''high'' neighbor sites and therefore

$$\Omega_n\simeq 2d(\ell^d-n)\frac{n}{\ell^d}. \tag{8}$$

The distribution of ''high'' sites is actually not random but we have verified numerically that, for large enough dimensions, Eq. (8) provides a reasonable approximation [23]. Combining Eqs. (7) and (8) one easily obtains the probability ρ_n of state n in the stationary state of the master equation

$$\rho_n=\rho_0\frac{r(0\rightarrow 1)}{r(n\rightarrow n+1)},\qquad n=1,\ldots,\ell^d-1, \tag{9}$$

where ρ_0 is fixed by the normalization condition $\Sigma_{n=0}^{\ell^d-1}\rho_n=1$. A simple calculation leads to

$$\rho_0=\left\{1+\frac{\ell^d}{2dx_k}\left[2d\ln\ell+\ln\left(\frac{1+2dx_k}{\ell^d+2dx_k}\right)\right]\right\}^{-1}. \tag{10}$$

The roughness of configurations with n particles, using Eq. (2), is $(1-n/\ell^d)n/\ell^d$ which, averaged over the distribution ρ_n [as specified by Eq. (9) and (10)] gives

$$\omega^2(\ell,x)\cong\rho_0\frac{\ell^d}{2dx_k} \tag{11}$$

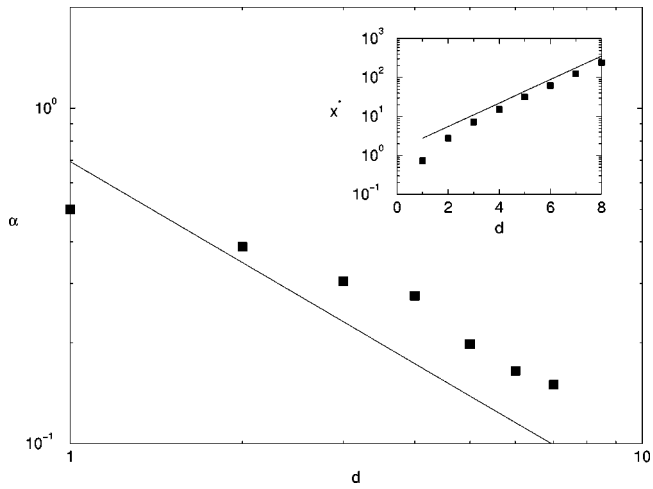

FIG. 2. Value of α from the present calculation (solid line) and from simulations of Refs. [8] and [18] for $d=1,\ldots,7$. In view of the approximations involved (see text) we obtain a lower bound. Inset: fixed point value $x^*(d)$ vs d. The theoretical prediction (solid line) is an upper bound to the true $x^*(d)$.

where we have assumed $dx_k\gg 1$ and $\ell^d\gg 1$. Combining Eq. (11) with the RG equation (5) we obtain, to leading order in d, a fixed point

$$x^*=2^{d+1}\ln 2. \tag{12}$$

consistent with the assumption $dx_k\gg 1$. Using now Eq. (6) it is straightforward to find, to leading order in $d\gg 1$,

$$\alpha\simeq\frac{1}{3(\ln 2)^2}\frac{1}{d}. \tag{13}$$

Furthermore, we can also analyze the stability of the fixed point. The derivative of the RG transformation $x_{k+1}=R(x_k)$ of Eq. (5), at the fixed point, is

$$R'(x^*)=-1+\frac{1}{2\ln 2}\frac{1}{d}+O(d^{-2}). \tag{14}$$

Since $|R'(x^*)|<1$ we can conclude that the fixed point is attractive $\forall d$. Therefore we find a finite, stable fixed point x^* with an exponent $\alpha>0$ for all d, which is the main result of this Rapid Communication. This excludes the occurrence of a finite upper critical dimension, d_c, which would show up, in the present framework, in a stable fixed point at $x^*=\infty$ for $d\geq d_c$.

Let us now discuss the validity of the approximations used. We neglected configurations with $h(i)\geq h_0+2$ or equivalently deposition processes on a ''high'' site. The rate of this process, on a state with n ''high'' sites, is $r_{\rm up}(n)=n$. Our approximation is then valid if $r(n\rightarrow n+1)\gg r_{\rm up}(n)$. This condition fails when the process is close to complete a new layer, i.e., for $n\simeq\ell^d$. More precisely the deposition on ''high'' sites is not important for

$$\ell^d-n\geq 1\gg\frac{\ell^d}{2dx}. \tag{15}$$

Since $x^* \sim 2^d$, the approximation is correct for $\ell = 2$ $\forall d$. Figure 1 shows that the approximation to the left-hand side (LHS) of Eq. (5) $1 + 4\omega^2(\ell = 2, x)$ is good already for $d = 7$. The approximation is much less accurate for $\ell = 4$ and, as a consequence, fluctuations in the system of size $\ell = 4$ are underestimated. This means that our approach underestimates the right-hand side (RHS) of Eq. (5) and consequently also its value at the intersection point with the LHS. This value is directly related to the roughness exponent by Eq. (6) and therefore *the restriction of height fluctuations leads to a lower bound to the exponent* α. Accordingly since the LHS of Eq. (5) decreases with x, Eq. (12) gives an upper bound to the true fixed point parameter x^*. Figure 1 illustrates this analysis for $d = 7$. Figure 2 shows a comparison of the present analytical estimates [Eqs. (12) and (13)] and the results of Refs. [8,18,19].

Besides the approximations of the present calculation, which, as we have argued, provide a lower bound to α, it is also worth discussing the approximations of the RG method itself. In this respect we observe that Eq. (3) is a minimal parametrization of the dynamics, in the sense that it allows for the minimal proliferation in the RG capturing the relevant features of KPZ growth. In principle, more proliferation parameters can be included in order to improve the accuracy of the method. It is important to note, however, that the range of typical fluctuations $h(i) - h(j) \sim \ell^\alpha$ allowed in the RG calculation is small and the one-parameter approximation in Eq. (3) to the scale invariant dynamics is reasonable. This is confirmed by the accuracy of the RG predictions in finite dimensions [19] and it is expected to improve as $\alpha \to 0$. Therefore the inclusion of additional proliferation parameters in Eq. (3) is not expected to change the nature of the fixed point and of our main conclusions. Let us also point out that usually small cells analysis becomes very accurate in high dimensions. An extension of the RG procedure to cells of larger size, going beyond the present approximations, provides in principle a systematic way to improve our prediction which is currently under investigation [22].

In conclusion, we have shown that the recently proposed [19] real space RG predicts that the roughness exponent α decreases not faster than $1/d$ as $d \to \infty$ [Eq. (13)]. This implies that there is no finite upper critical dimension in the KPZ universality class and it suggests that theoretical arguments leading to $d_c = 4$ should be reconsidered.

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